Shear Formula for Beams of Variable Cross Section

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THE formula for evaluating the shearing stress in a beam of constant cross section is well known. The derivation† of this formula may be found in most strength-of-materials undergraduate texts.[1]

Proceeding in the same manner as for beams of constant cross section,[1] the formula

$$\tau_{xy} = \frac{I}{b_{(y)}} \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{MQ}{I} \right] \tag{1}$$

may be derived for beams of variable cross section. The assumptions and limitation in Eq. (1) are the same as those for constant cross sections and the quantities M, Q, I, and b have the same definition as those in Ref. 1. Equation (1) is valid for beams of variable height and váriable thickness (such as tapered I beams), and reduces to the Zuravski formula for constant cross section since then Q and I are constant. Solutions using Eq. (1) will now be compared to those exact solutions obtained by theory of elasticity.

Solution 1

Consider bending of the wedge shown in Fig. 1. Equation (1) in this case yields

$$\tau_{xy} = -Px^2/I \tag{2}$$

An exact elasticity solution of this problem (for a small angle α) derived in Ref. 2, p. 111, Eq. (c), is

$$\tau_{xy} = -Px^2 [(\tan\alpha/\alpha)^3 \sin^4\theta]/I$$

Equation (2) compares well with the exact solution for a small wedge angle 2α. Table 1 compares the ratios of the shearing stress (on the inclined surface) for increasing semi-wedge angle. Equation (1) is within 13% of the exact solution for wedges with $\leq 20°$ or for beams with aspect ratios $(\ell/h) \geq 1.37$.

Solution 2

The problem of the bending of the wedge of Fig. 1 may also be solved (for σ_r), when the thickness is a polynominal function of the radial coordinate r, using the method of Ref. 3. The solution for the vertical shearing stress for small wedge angles α and thickness $t = t_0 r$ is

$$\tau_{xy} = -\frac{3}{2} \frac{P}{t_0} \frac{x^2}{x_0^4} \tan\alpha \left[\frac{\tan\alpha}{\alpha} \right]^3 \cos^5\theta$$

where $x_0 = y \tan\alpha$. Equation (1) yields in this case

$$\tau_{xy} = -3Px^2 \tan\alpha/(2t_0 x_0^4)$$

again indicating that this solution is valid for a narrow wedge with a linearly varying height as well as a linearly varying thickness.

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†This formula was first developed in Russia in 1855 by D.I. Zuravski.

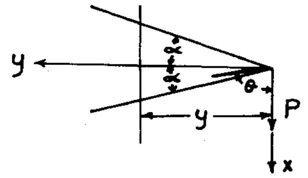

Fig. 1 Bending of a wedge by a force applied at the end.

Table 1 Comparison of the ratio of τ_{xy} from Eq. (2) and Eq. (c) of Ref. 2.

α (deg)	1	5	10	15	20	25	30
% error	0.03	0.77	3.1	7.1	13.1	21.4	32.6

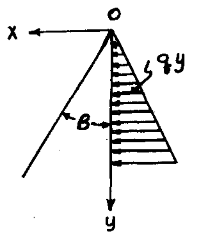

Fig. 2 Retaining wall subjected to the pressure of water.

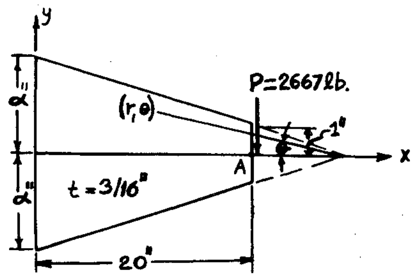

Fig. 3 Tapered beam used to compare elasticity and Eq. (1) solutions.

Table 2 Comparison of elasticity and strength of material solutions for shearing stress in a narrow and a widely tapered beam of Fig. 3

Coordinate	Locations	Narrow wedge ($\alpha = 3$)		Coordinate	Locations	Wide wedge ($\alpha = 20$)	
x	y	Eq. (1)	Eq. (3)	x	y	Eq. (1)	Eq. (3)
2	0	1360	1366	2	0	32.6	42.5
2	0.93	1753	1758	2	6.03	152.7	249.6
2	1.87	2933	2932	2	12.07	513.1	564.9
2	2.8	4898	4880	2	18.1	1113.7	691.4
10	0	2667	2678	10	0	96.8	126.3
10	0.67	2963	2967	10	3.5	290.3	445.3
10	1.33	3852	3845	10	7.0	870.9	947.7
10	2.0	5334	5333	10	10.5	1838	1162.7
at point A		10,688	10,711	at point A		10,668	13,940

Solution 3

Consider next bending of the retaining wall of Fig. 2. The quantities entering Eq. (1) for this case are

$$M = qy^3/6 \quad I = (y \tan\beta)^3/12 \quad Q = x(y \tan\beta - x)/2$$

and hence

$$\tau_{xy} = \mathrm{d}/\mathrm{d}y \; [qx(y\tan\beta - x)/\tan^3\beta] = qx/\tan^2\beta$$

The exact elasticity solution of this problem, Eq. (6.75) of Ref. 4, yields the same solution.

The authors in Refs. 2 and 4 should not compare their exact solutions with the Zuravski formula, since this formula is not applicable for beams of variable cross section. The present author shows that a comparison of the solutions in Refs. 2 and 4 with those from Eq. (1) results in satisfactory agreement. These solutions could also be compared using the method derived in Ref. 5 for beams of variable height and constant thickness.

The tapered beam of constant thickness shown in Fig. 3 will now be analyzed for shearing stress using theory of elasticity and strength of materials, Eq. (1). An elasticity solution to the beam problem of Fig. 3 may be obtained by a proper superposition of the solutions given for the beams of Figs. 64 and 65 in Ref. 2. Such a solution is

$$\tau_{xy}=\sigma_r \sin 2\theta/2+\tau_{r\theta}\cos 2\theta \quad \text{where } \sigma_\theta=0 \tag{3a}$$

$$\sigma_r=-\frac{2P\sin\theta}{r(2\alpha-\sin 2\alpha)}+\frac{2Pa\sin 2\theta}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)}$$

$$\tau_{r\theta}=-\frac{Pa(\cos 2\theta-\cos 2\alpha)}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)} \tag{3b}$$

A narrow wedge ($\alpha=3$) and a wide wedge ($\alpha=20$) of the tapered beam shown in Fig. 3 have been solved for the shearing stress at several points. The results calculated by theory of elasticity Eq. (3) and strength of materials Eq. (1) are summarized in Table 2. The maximum shearing stress, which occurs at point A (Fig. 3) is also included. These results again indicate that Eq. (1) may be used for the narrow tapered beam. Since solutions obtained from Eq. (1) may be used for tapered beams with small wedge angles, it appears that Eq. (1) may also be used for beams of various other shapes and thickness.

References

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Peak Distributions of Random Response Processes

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It has been recognized[1] that, from the standpoint of the strength of a linear or nonlinear structure, it is more important to study the stochastic distribution of the peak and the highest peak of the response, rather than that of the response itself, when systems are analyzed against random excitations of the stationary or nonstationary type.

A peak or a maximum in a sample random function $x(t)$ of a continuously valued random process if also continuous with time t and occurs when $\dot{x}(t)$ is zero and $\ddot{x}(t)$ is negative, where $x(t)$ represents the response of the system, single or multidegree of freedom, maybe stress, displacement, or strain at a critical point or zone and $\dot{x}(t)$, $\ddot{x}(t)$ its first and second derivatives, respectively.

In a large class of problems, the designer will only be interested in estimating the distribution of the largest of the maximum occurring within a specified period, and the distribution of the peaks may be of little concern to him.

Standard methods to estimate the distribution of the peaks making use of the joint probability density function of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ involve tedious numerical calculations.[2] However, no exact procedure is available for the evaluation of the distribution of the largest of the peaks. An approximate solution to these problems is attempted in this study.

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Analysis

If $x(t)$ represents a nonstationary random process, say the response of a single degree of freedom system, the number of extrema in $x(t)$, $\epsilon(\lambda, t_1, t_2)$ above a specified level λ within a time interval (t_1, t_2) can be expressed as[2]

$$\epsilon(\lambda,t_1,t_2)=\int_{t1}^{t2}|\ddot{x}(t)|\delta[\dot{x}(t)]\mathbf{1}[x(t)-\lambda]\,dt \tag{1}$$

where $\mathbf{1}$ [ ] represents Heaviside's step function and δ [ ] Dirac's delta function.

Following Rice,[3] the expected number of times a level is crossed from below in an interval (t_1, t_2) is given by

$$E[N(\lambda,t_1,t_2)]=\int_{t1}^{t2}\int_0^\alpha |\dot{x}|p(\lambda,\dot{x};t)\,d\dot{x}dt \tag{2}$$

where $p(x, \dot{x}; t)$ is the joint function of $x(t)$ and $\dot{x}(t)$ given by

$$p(x,\dot{x},t)=\frac{1}{2\pi\sigma_1\sigma_2(1-p^2)^{1/2}}\exp\left[-\frac{1}{2(1-p^2)}\left\{\left[\frac{x}{\sigma_1}\right]^2-\frac{2px\dot{x}}{\sigma_1\sigma_2}+\left[\frac{\dot{x}}{\sigma_2}\right]^2\right\}\right] \tag{3}$$

where $p(t)$ is the correlation function of $x(t)$ and $\dot{x}(t)$; $\sigma_1(t)$ and $\sigma_2(t)$ are the standard deviations of $x(t)$ and $\dot{x}(t)$, respectively. The expected number of peaks above the zero level of response can then be estimated using Eq. (2) with $\lambda=0$, assuming that there is only one peak associated with the response process crossing this level. The probability distribution function of the peaks at the level λ, F_p (λ, t) is then

$$F_p(\lambda,t)=\frac{\int_{t1}^{t2}\int_0^\alpha|\dot{x}|p(\lambda,\dot{x};t)d\dot{x}dt}{\int_{t1}^{t2}\int_0^\alpha|\dot{x}|p(0,\dot{x};t)\,d\dot{x}dt} \tag{4}$$

For estimating the probability density and distribution of the largest of the maxima, the results obtained by Davenport[4] are used. The probability $p_L(\lambda,t)$ that the largest of the peaks has a value λ is that one of the maxima has this value and the rest are smaller within the interval considered. The required probability can thus be expressed as:

$$p_L(\lambda,t)=N[1-F_p(\lambda,t)]^{N-1}dF_p(\lambda,t)/d\lambda \tag{5}$$

Where N is the expected number of peaks in the interval (t_1, t_2) above the zero level and is given by Eq. (2) with $\lambda=0$. The statistical properties of the distribution can be easily estimated from Eq. (5). For example, the mean of the largest of the maxima, L (λ,t) is given by

$$L(\lambda,t)=\int_0^\alpha p_L(\lambda,t)\lambda d\lambda \tag{6}$$